1 Research Statement

During my PhD thesis, I was interested in optimal transportation on Wiener spaces. There are mainly three natural norms or pseudo-norms on the classical Wiener space (X, H, μ).
We consider often the classical Wiener space which is $X = \mathcal{C}([0,1], \mathbb{R})$ endowed with the supremum norm, defined by

$$\|x\|_\infty := \sup_{t\in[0,1]} |x(t)|.$$

$(X, \|.\|_\infty)$ is a separable Banach space and has a Hilbert subspace dense, which is called the Cameron-Martin space. The natural Cameron-Martin pseudo-norm associated is the following:

$$|x|_H := \left(\int_0^1 |\dot{x}(t)|^2 dt\right)^{1/2}$$

which is finite only when x belongs to the Cameron-Martin space H. It turns out that $\mu(H) = 0$, so that the previous norm is infinite $\mu-$almost everywhere.
We can endow X with an other norm, defined as:

$$\|x\|_{k,\gamma} := \left(\int_0^1 \int_0^1 \frac{(x(t) - x(s))^{2k}}{|t-s|^{1+2k\gamma}} dtds\right)^{1/2k}$$

for suitable parameters k, γ. This norm has been introduced by Airault and Malliavin in [1]. The subset of $(\tilde{X} := \{x \in X, \|x\|_{k,\gamma} < \infty\}, \|.\|_{k,\gamma})$ is also a separable Banach space and we still have $\mu(\tilde{X}) = 1$.
There is a relation between these norms, that is:

$$\|x\|_\infty \leq \|x\|_{k,\gamma} \leq C_{k,\gamma}|x|_H \quad \text{for any } x \in X. \tag{1.1}$$

The purpose of my PhD thesis is to study optimal transportation with respect to these different norms.

Many of my results are gathered in [13] and [9]. They concern:

1.1 the $K-$convexity of relative entropy

The study of convexity of relative entropy brings us some information of the geometry of the ambiant space, in particular in term of *lower bound of Ricci curvature*. Indeed Sturm and von Renesse proved in [15] that on a smooth Riemannian manifold, there is characterizations of uniform lower bounds for the Ricci curvature in terms of convexity properties of the relative entropy.
Their work was the starting point which motivates mathematicians to introduce a kind of Ricci curvature over general metric measure spaces. In fact we can always study convexity properties of the relative entropy over metric measure spaces. And this is the way we suggest developing here.
The *relative entropy w.r.t.* μ is defined as:

$$Ent_\mu(\rho) := \begin{cases} \int f\log(f)d\mu & \text{if } \rho \text{ admits } f \text{ for density w.r.t } \mu \\ +\infty & \text{otherwise} \end{cases} \tag{1.2}$$

The space of probability measures $\mathcal{P}(X)$ over X, can be seen as a metric space when it is endowed by the so-called $p-$ *Wasserstein distance* W_p (provided the measures have finite $p-$moments). Notice that the distance W_p is induced by the distance on X, namely it is not the same when we look X endowed with one of the three norm defined above.
Thus we can define a notion of (constant speed) *geodesics* on this metric space. It turns out that in our cases, the relative entropy is convex along such geodesics. This is called the *displacement convexity*.

Theorem 1.1. *Let $p \in [1,2]$. If ρ_0 and ρ_1 are probability measures on X both absolutely continuous with respect to μ, then there exists some (constant speed) geodesic ρ_t (w.r.t. the distance W_p) such that:*

$$Ent_\mu(\rho_t) \leq (1-t)Ent_\mu(\rho_0) + tEnt_\mu(\rho_1) - \frac{Kt(1-t)}{2} W_p^2(\rho_0, \rho_1) \qquad \forall t \in [0,1].$$

Whatever $p \in [1,2]$, it turns out that for the infinite norm, K is equal to 1, and that for the Sobolev-type norm $\|.\|_{k,\gamma}$, K is equal to $1/C_{k,\gamma}^2$.
To get this kind of result we process by finite dimensional approximation as Fang, Shao and Sturm in [8], who have treated the case of the Cameron-Martin norm. Our main contribution consists in establishing results without applying powerfull tools like *Gromov-Hausdorff* convergence (see [12]) or $\mathbb{D}-$convergence introduced by Sturm in [14].

1.2 the Monge Problem

Let ρ_0 and ρ_1 be two probability measures on X. The *Monge Problem* consists in finding a Borel map T which pushes ρ_0 forward to ρ_1 and minimizing the quantity:

$$\int_X c(x, T(x))d\rho_0(x), \tag{1.3}$$

where $c(x,y)$ is called a cost function.
The following result, which solves the Monge Problem when the cost is induced by the norm $\|.\|_{k,\gamma}$ to the power of p, can be found in my first paper [13]. It is based on the paper [7] of Fathi and Figalli.

Theorem 1.2.
1. *If $p > 1$ and ρ_0 is absolutely continuous with respect to μ, then there exists a map T (unique up to a set of zero measure for ρ_0), which minimizes (1.3). Moreover there is a unique optimal transference plan between ρ_0 and ρ_1 relatively to the cost $\|.\|_{k,\gamma}^p$, which is exactly $(Id \times T)_\# \rho_0$.*

2. *If $p = 1$ and both ρ_0 and ρ_1 are absolutely continuous with respect to μ, then there exists an optimal transference plan Π between ρ_0 and ρ_1 relatively to the cost $\|.\|_{k,\gamma}$, such that Π is concentrated on a graph of some map T minimizing (1.3).*

It turns out that these norms depending on two parameters converge to a norm equivalent to the uniform norm. We could expect the sequence of corresponding optimal maps to converge to one for a suitable cost function.

With S. Fang we prove an extension of the result of Feyel and Üstünel in [10], which concerns the Monge Problem relatively to the Cameron-Martin cost. It improves their result in the sense that we do not require a selection theorem.
Assume that our measures have both densities, say $\rho_0 = e^{-V} d\mu$ and $\rho_1 = e^{-W} d\mu$ with

$$\int_X |\nabla V|^2 e^{-V} d\mu < +\infty, \tag{1.4}$$

and $W \in \mathbb{D}_2^2(X)$ such that e^{-W} is bounded, satisfies:

$$\nabla^2 W \geq -c \,\mathrm{Id}, \quad c \in [0,1[. \tag{1.5}$$

Under above conditions on V and W we have the result:

Theorem 1.3. *There is a $\psi \in \mathbb{D}_1^2(X, e^{-W}\mu)$ such that $x \to S(x) = x + \nabla\psi(x)$ is the optimal transport map which pushes $e^{-W}\mu$ to $e^{-V}\mu$; moreover the inverse map of S is given by $x \to x + \eta(x)$ with $\eta \in L^2(X, H; e^{-V}\mu)$.*

An essential goal is to obtain similar result when the cost is induced by the uniform norm.

1.3 the regularity of optimal maps

In the paper written with S. Fang [9], we were interested in obtain regularity for the optimal map in $\mathbb{R}^d$, when the cost is the square of the Euclidian norm. The probability measures are taken absolutely continuous with respect to the standard Gaussian measure γ, whose densities have full support $\mathbb{R}^d$, with the form e^{-V} and e^{-W}. The choice of the reference measure is made in order to extend these results in Wiener spaces.

Our work has been motivated by those of Bogachev and Kolesnikov: [3] and [4].
Let us consider two optimal maps:

$$(T_1)_\# : e^{-V_1}\gamma \longrightarrow e^{-W_1}\gamma,$$
$$(T_2)_\# : e^{-V_2}\gamma \longrightarrow e^{-W_2}\gamma,$$

The approach is based on derivative of corresponding Monge-Ampère equations.

Suppose $W_1, W_2 \in \mathbb{D}_1^2(\mathbb{R}^d, \gamma), W_1 \in \mathbb{D}_2^2(\mathbb{R}^d, \gamma)$ verify

$$\nabla^2 W_1 \geq -cId \quad \text{with} \quad c < 1.$$

Theorem 1.4. *It holds :*

$$\int_{\mathbb{R}^d} |T_1 - T_2|^2 e^{-V_2} d\gamma \leq \frac{4}{1-c} Ent_{e^{-V_1}\gamma}(e^{V_1 - V_2}) + \frac{4}{(1-c)^2} \int_{\mathbb{R}^d} |\nabla(W_1 - W_2)|^2 e^{-W_2} d\gamma$$

It is worth to notice that if $W_2 = W_1$ then the above inequality gives:

$$\int_{\mathbb{R}^d} |T_1 - T_2|^2 e^{-V_2} d\gamma \leq \frac{4}{1-c} Ent_{e^{-V_1}\gamma}(e^{V_1 - V_2}).$$

The previous inequalit generalizes the famous Talagrand's inequality.

Now suppose $V \in \mathbb{D}_1^2(\mathbb{R}^d, \gamma), W \in \mathbb{D}_2^2(\mathbb{R}^d, \gamma)$ verify

$$e^{-V} \leq \delta_2, \quad e^{-W} \leq \delta_2, \quad \nabla^2 W \geq -cId \quad \text{with} \quad c \in [0, 1). \tag{1.6}$$

A crucial result of our purpose is the following dimension-free inequality:

Theorem 1.5.

$$\begin{aligned} &\int_{\mathbb{R}^d} |\nabla V|^2 e^{-V} d\gamma - \int_{\mathbb{R}^d} |\nabla W|^2 e^{-W} d\gamma + \frac{2}{1-c} \int_{\mathbb{R}^d} ||\nabla^2 W||_{HS}^2 \, e^{-W} d\gamma \\ &\geq 2\text{Ent}_\gamma(e^{-V}) - 2\text{Ent}_\gamma(e^{-W}) + \frac{1-c}{2} \int_{\mathbb{R}^d} ||\nabla^2 \varphi||_{HS}^2 \, e^{-V} d\gamma. \end{aligned} \tag{1.7}$$

1.4 Monge-Ampère equations

The first result concerns Monge-Ampère equation in Sobolev spaces. Let $V \in \mathbb{D}_1^2(\mathbb{R}^d, \gamma)$ and $W \in \mathbb{D}_2^2(\mathbb{R}^d, \gamma)$ satisfying conditions (1.6).

Theorem 1.6. *The optimal transport map $x \to x + \nabla\varphi(x)$ from $e^{-V}\gamma$ to $e^{-W}\gamma$ solves the following Monge-Ampère equation*

$$e^{-V} = e^{-W(\nabla\Phi)} e^{\mathcal{L}\varphi - \frac{1}{2}|\nabla\varphi|^2} \det{}_2(\text{Id} + \nabla^2\varphi),$$

where $\nabla\Phi(x) = x + \nabla\varphi(x)$.

Now we focus on an abstract Wiener space (X, H, μ).
Let $V \in \mathbb{D}_1^2(X,\mu)$ and $W \in \mathbb{D}_2^2(X,\mu)$. Assume that (1.6) holds and in addition that

$$e^{-V} \geq \delta_1 > 0.$$

It turns out that on the basis of above theorem, and combining with the dimension-free inequality 1.5, we can increase the dimension and get similar result in infinite dimension.

Theorem 1.7. *There exists a function $\varphi \in \mathbb{D}_2^2(X)$ such that $x \to x + \nabla\varphi(x)$ pushes $e^{-V}\mu$ to $e^{-W}\mu$ and solves the Monge-Ampère equation*

$$e^{-V} = e^{-W(T)} e^{\mathcal{L}\varphi - \frac{1}{2}|\nabla\varphi|^2}\, det_2(\mathrm{Id}_{H\otimes H} + \nabla^2\varphi),$$

where $T(x) = x + \nabla\varphi(x)$.

This previous result includes two special cases:

- when the source measure is the Wiener measure, by Feyel and Üstünel in [11];
- when the target measure is the Wiener measure, by Bogachev and Kolesnikov in [4].

2 Projects

This last period of my PhD thesis, I intend to finish the resolution of Monge Problem in the classical Wiener space when the cost is induced by the uniform norm. With the support of T. Champion and L. De Pascale, I am deeply convinced that this work will succeed.

As regards the regularity of the optimal map for the square of the Cameron-Martin norm, an interesting question is to determine the regularity of the skeleton of optimal map, namely to know whether the optimal map would be "approximately continuous" along the Cameron-Martin space. This notion of approximately continuous is treated in $\mathbb{R}^d$ in the book of Ambrosio, Gigli and Savaré [2]. Some dimension-free inequalities proved in my second paper (with S. Fang) [9] could be very useful for this purpose.

For the sequel I would like to publish my PhD thesis in a reference book, concerning optimal transportation in Wiener spaces.

References

[1] H. Airault P. Malliavin, Integration geometrique sur l'espace de Wiener, *Bulletin des Sciences Mathematiques*, 1988, 3-52.

[2] L. Ambrosio, N. Gigli and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lect. in Math., ETH Zürich, Birkhäuser Verlag, Basel, 2005.

[3] V.I. Bogachev and A.V. Kolesnikov, On the Monge-Ampère equation in infinite dimensions, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 8(2005), 547572.

[4] V.I. Bogachev and A.V. Kolesnikov, Sobolev regularity for the Monge-Ampère equation in the Wiener space, *arXiv:1110.1822v2.*

[5] F. Cavalletti, F, The Monge Problem in Wiener space, *Calculus of Variations*, 2011.

[6] T. Champion and L. De Pascale, Monge Problem in $\mathbb{R}^d$, *Duke Math. J.*, 157(2011), 551-572.

[7] A. Fathi and A. Figalli, Optimal transportation on non-compact manifolds, *Israel J. Math.*, 2010, 1-59.

[8] S. Fang, J. Shao and K-T. Sturm, Wasserstein space over the Wiener space, *Probab. Theory Related Fields*, 2010, 535-565.

[9] S. Fang and V. Nolot, Sobolev estimates for optimal transport maps on Gaussian spaces, *arXiv:1207.4907.*

[10] D. Feyel and A.S. Üstünel, Monge-Kantorovich measure transportation and Monge-Ampère equation on Wiener space, *Prob. Th. related Fields*, 128(2004), 347-385.

[11] D. Feyel and A.S. Üstünel, Solution of the Monge-Ampère equation on Wiener space for general log-concave measures, *J. Funct. Anal.*, 232(2006), 29-55.

[12] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, *Annals of Mathematics*, 2009, 903-991.

[13] V. Nolot, Optimal transport on Wiener space with different norms, *arXiv:1111.1938.*

[14] K-T. Sturm, On the Geometry of Metric Measure Spaces I., *Acta Math*, 2006, 65-131.

[15] K-T. Sturm and M-K. v Renesse, Transport Inequalities, Gradient Estimates, Entropy and Ricci Curvature, *Pure Appl. Math.*, 58(2005), 923-940.